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Supersymmetry and spin systems

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Abstract. It is shown that supersymmetry may be applied to spin systems. A simple algebra is proposed and various examples are discussed. It is argued that certain correlation functions must vanish on account of supersymmetry.

1. Introduction

Supersymmetry has been a subject of investigation among elementary particle theorists for some time (Wess and Zumino 1974, Ferrara *et al* 1974, Wess 1974 and references therein); however, until now, no use seems to have been made of it in the realm of statistical mechanics. It is our intention to close this gap†.

The basic ingredient of supersymmetry is the use of anticommutators as well as commutators in the algebra of symmetry generators; in relativistic quantum field theory this has so far proved to be the only way to circumvent certain 'no-go theorems' concerning the non-trivial fusion of internal symmetries and Poincaré invariance (Coleman and Mandula 1967). The algebra of sypersymmetry is not an ordinary Lie algebra‡; only by the use of totally anticommuting parameters (elements of a Grassmann algebra) can it be integrated to a group of symmetry transformations.

Supersymmetry has been realized in the framework of Lagrangian field theory where it leads to remarkable and unprecedented consequences. (i) The conserved charges are spinorial rather than scalar (this might provide a profound reason for the masslessness of the neutrino). (ii) Supersymmetric theories are far less divergent than any other known field theory (such as quantum electrodynamics). It is to be expected that an application of supersymmetry to statistical mechanics will have similar far-reaching consequences.

Our approach will differ somewhat from the one described: the algebra that we propose is non-relativistic as one may not expect the relativistic algebra to be of much value for the description of a non-relativistic lattice—in fact, our algebra will be simpler than its relativistic counterpart. Also we define supersymmetry in the Hamiltonian formalism, which is probably more familiar to statistical physicists, before we turn to the Lagrangian formalism which seems to be more popular among field theorists. Another difference is the occurrence of pure spin–spin interactions in our non-relativistic approach.

For simplicity we shall only discuss the one-dimensional chain lattice and merely indicate the generalization to an arbitrary-dimensional lattice as the generalization will

† The following review is necessarily somewhat impressionistic.
‡ It is sometimes called a 'pseudo-Lie algebra'.

be obvious in all cases. Furthermore, we shall always assume periodic boundary conditions.

For later convenience we fix our notation: $j, k, l, \ldots \in \{1, \ldots, N\}$ set of indices; a_j, a_j^* are annihilation and creation operators of Bose excitations at j; ψ_j, $\bar{\psi}_j$ are annihilation and creation operators of Fermi excitations at j; $\zeta, \zeta', \ldots$ are totally anticommuting parameters.

We then have the following commutation and anticommutation relations:

$$\begin{aligned}
&[a_j, a_k] = [a_j^*, a_k^*] = 0; \qquad [a_j, a_k^*] = \delta_{jk} \\
&\{\psi_j, \psi_k\} = \{\bar{\psi}_j, \bar{\psi}_k\} = 0; \qquad \{\psi_j, \bar{\psi}_k\} = \delta_{jk} \\
&[a_j, \psi_k] = [a_j^*, \psi_k] = [a_j, \bar{\psi}_k] = [a_j^*, \bar{\psi}_k] = 0 \\
&\{\zeta, \zeta'\} = 0; \qquad \{\zeta, \psi_j\} = \{\zeta, \bar{\psi}_j\} = 0 \\
&[\zeta, a_j] = [\zeta, a_j^*] = 0.
\end{aligned} \tag{1.1}$$

Furthermore

$$a_{j+N} \equiv a_j, \; a_j^* \equiv a_{j+N}^*; \qquad \psi_j \equiv \psi_{j+N}, \; \bar{\psi}_j \equiv \bar{\psi}_{j+N}. \tag{1.2}$$

We shall use anticommuting Fermi operators consistently throughout this paper; nonetheless we occasionally give the results in usual notation using Pauli matrices.

The connection between the two descriptions is explained in the appendix.

2. Basic algebra and a very simple model

The basic algebra is

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0; \qquad \{Q, Q^\dagger\} = H \tag{2.1}$$

$$[Q, H] = [Q^\dagger, H] = 0. \tag{2.2}$$

Q, $Q^\dagger$, are the generators of supersymmetry transformations and H will be taken as the Hamiltonian of the system (it is understood that the operators act on some Hilbert space $\mathscr{H}$). It should be noted that (2.2) is a consequence of (2.1) and expresses the invariance of H under supersymmetry transformations. Using anticommuting parameters ζ, ζ', (2.1) and (2.2) become

$$[\zeta Q, \zeta' Q] = [Q^\dagger \bar{\zeta}, Q^\dagger \bar{\zeta}'] = 0; \qquad [\zeta Q, Q^\dagger \bar{\zeta}] = \zeta\bar{\zeta} H \tag{2.3}$$

$$[\zeta Q + Q^\dagger \bar{\zeta}, H] = 0\dagger. \tag{2.4}$$

A group element g may be written as

$$g = \exp(\mathrm{i}\zeta Q + \mathrm{i} Q^\dagger \bar{\zeta} + \mathrm{i} t H), \qquad t \in \mathbb{R} \tag{2.5}$$

and from this using the Hausdorf formula and (2.3), one obtains

$$\begin{aligned}
gg' &= \exp[\mathrm{i}(\zeta Q + Q^\dagger \bar{\zeta} + tH)] \exp[\mathrm{i}(\zeta' Q + Q^\dagger \bar{\zeta}' + t'H)] \\
&= \exp[\![\mathrm{i}\{(\zeta + \zeta')Q + Q^\dagger(\bar{\zeta} + \bar{\zeta}') + [t + t' - \tfrac{1}{2}\mathrm{i}(\zeta\bar{\zeta}' - \zeta'\bar{\zeta})]H\}]\!].
\end{aligned} \tag{2.6}$$

If one used commuting instead of anticommuting parameters it would be impossible to obtain a closed expression for gg' in terms of Q, $Q^\dagger$, H due to the unknown value of

† $\{Q, \zeta\} = \{Q^\dagger, \zeta\} = 0$.

$[Q, Q^\dagger]$ Therefore the use of anticommuting parameters is essential. If A is any operator the (canonically) transformed operator is

$$A' = e^{-i(\zeta Q + Q^\dagger \bar{\zeta})} A \, e^{i(\zeta Q + Q^\dagger \bar{\zeta})} \tag{2.7}$$

and the infinitesimal† transformation $\delta A = A' - A$ is given by

$$\delta A = i^{-1}[\zeta Q + Q^\dagger \bar{\zeta}, A]. \tag{2.8}$$

The simplest realization of (2.1), (2.2) is provided by

$$Q := \sum_{j=1}^{N} a_j^* \psi_{j+l} \qquad Q^\dagger = \sum_{j=1}^{N} \bar{\psi}_{j+l} a_j \qquad (l = 0, \ldots, N-1) \tag{2.9}$$

$$H := \sum_{j=1}^{N} (a_j^* a_j + \bar{\psi}_j \psi_j). \tag{2.10}$$

Proof.

$$\{Q, Q\} = \sum_{j,k=1}^{N} \{a_j^* \psi_{j+l}, a_k^* \psi_{k+l}\} = \{Q^\dagger, Q^\dagger\}^\dagger = 0$$

$$\begin{aligned} \{Q, Q^\dagger\} &= \sum_{j,k=1}^{N} \{a_j^* \psi_{j+l}, \bar{\psi}_{k+l} a_k\} \\ &= \sum_{j,k=1}^{N} \{[a_j^*, a_k] \psi_{j+l} \bar{\psi}_{k+l} + a_k a_j^* \{\psi_{j+l}, \bar{\psi}_{k+l}\}\} \\ &= \sum_{j=1}^{N} (-\psi_{j+l} \bar{\psi}_{j+l} + a_j a_j^*) = \sum_{j=1}^{N} (a_j^* a_j + \bar{\psi}_j \psi_j). \end{aligned} \tag{2.11}$$

The variation of a_j, a_j^*, ψ_j, $\bar{\psi}_j$ is given by

$$\begin{aligned} \delta a_k &= i^{-1}[\zeta Q + Q^\dagger \bar{\zeta}, a_k] = -i\zeta \sum_{j=1}^{N} [a_j^* \psi_{j+l}, a_k] = i\zeta \psi_{k+l} \\ \delta a_k^* &= i^{-1}[\zeta Q + Q^\dagger \bar{\zeta}, a_k^*] = -i \sum_{j=1}^{N} [\bar{\psi}_{j+l} a_j, a_k^*] \bar{\zeta} = -i \bar{\psi}_{k+l} \bar{\zeta} \\ \delta \psi_k &= i^{-1}[\zeta Q + Q^\dagger \bar{\zeta}, \psi_k] = +i \sum_{j=1}^{N} \{\bar{\psi}_{j+l} a_j, \psi_j\} \bar{\zeta} = i a_{k-l} \bar{\zeta} \\ \delta \bar{\psi}_k &= i^{-1}[\zeta Q + Q^\dagger \bar{\zeta}, \bar{\psi}_k] = -i\zeta \sum_{j=1}^{N} \{a_j^* \psi_{j+l}, \bar{\psi}_k\} = -i\zeta a_{k-l}^*. \end{aligned} \tag{2.12}$$

This set of infinitesimal transformations exhibits a characteristic feature of supersymmetry transformations: Fermi operators are transformed into Bose operators and vice versa. Or, loosely speaking, odd and even operators are transformed into one another.

† A topology on Grassmann algebras can be defined (Berezin 1966).

Remark 1. The generalization to a d-dimensional lattice is evident. Instead of (2.9) one could take

$$Q = \sum_{j\in G} \sum_{\hat{n}} a_j^* \psi_{j+l\hat{n}} \qquad Q^\dagger = \sum_{j\in G} \sum_{\hat{n}} \bar{\psi}_{j+l\hat{n}} a_j \tag{2.9'}$$

where $j \in G$ is now a multi-index assuming N^d values and $\hat{n}$ runs through the d unit vectors of the lattice.

Remark 2. Polarization of the a_j may be included by giving them an additional index $\alpha \in \{1, \ldots, p\}$:

$$Q_\alpha := \sum_{j=1}^{N} a_{j,\alpha}^* \psi_{j+l} \qquad Q_\alpha^\dagger = \sum_{j=1}^{N} \bar{\psi}_{j+l} a_{j,\alpha}. \tag{2.9''}$$

One may then postulate for example that Q_α transforms as a vector and thereby add a geometrical symmetry.

The Hamiltonian (2.10) describes a non-interacting system of N harmonic oscillators and N spins. The state-space $\mathscr{H}$ is spanned by all vectors of the form

$$|n_1, \sigma_1\rangle \otimes \ldots \otimes |n_N, \sigma_N\rangle =: |\{n_j\}, \{\sigma_j\}\rangle \dagger \tag{2.13}$$

where $n_j = 0, 1, 2, \ldots, \sigma_j = 0, 1$ and the energy of a given state is

$$\langle\{n_j\}, \{\sigma_j\}|H|\{n_j\}, \{\sigma_j\}\rangle = \sum_{j=1}^{N} (n_j + \sigma_j). \tag{2.14}$$

The eigenvalues of $Q + Q^\dagger$ may be found easily‡ ($|E, \lambda\rangle$ = any eigenstate of H, λ = degeneracy parameter):

$$\|(Q+Q^\dagger)|E,\lambda\rangle\|^2 = \langle E,\lambda|(Q+Q^\dagger)^2|E,\lambda\rangle = \langle E,\lambda|H|E,\lambda\rangle = E. \tag{2.15}$$

Therefore

$$\begin{aligned} (Q+Q^\dagger)|E,\lambda_+\rangle &= +\sqrt{E}|E,\lambda_+\rangle \\ (Q+Q^\dagger)|E,\lambda_-\rangle &= -\sqrt{E}|E,\lambda_-\rangle \end{aligned} \tag{2.16}$$

where any additional degeneracy is parametrized by $|0\rangle$ by λ_+, λ_-. Note that an Ising-like interaction may be constructed by taking $gH + g'H^2$ as the invariant Hamiltonian which, however, contains unphysical long-range interactions.

3. Supersymmetric spin system without Bose operators

To obtain a non-trivial interaction one has to go beyond the definition (2.9) and incorporate expressions which are trilinear in the operators. As Bose operators can be omitted, we consider only the case of an interacting system of spins which are aligned on a chain. To make the notation more transparent we assume that the chain has $2N$

† $\bar{\psi}|0\rangle = |1\rangle$, $\psi|1\rangle = |0\rangle$; $\psi|0\rangle = \bar{\psi}|1\rangle = 0$. $|0\rangle$ corresponds to 'spin down', $|1\rangle$ to 'spin up'.

‡ Q and $Q^\dagger$ are not Hermitian separately and therefore do not correspond to observables.

spins†. We then define:

$$Q := \sum_{j=1}^{N} \psi_{2j+1}\bar{\psi}_{2j}\psi_{2j-1} \qquad Q^{\dagger} = \sum_{j=1}^{N} \bar{\psi}_{2j-1}\psi_{2j}\bar{\psi}_{2j+1}. \tag{3.1}$$

The anticommutators are‡:

$$\begin{aligned}
\{Q, Q\} &= \sum_{j,k=1}^{N} \{\psi_{2j+1}\bar{\psi}_{2j}\psi_{2j-1}, \psi_{2k+1}\bar{\psi}_{2k}\psi_{2k-1}\} = 0 \\
\{Q^{\dagger}, Q^{\dagger}\} &= \sum_{j,k=1}^{N} \{\bar{\psi}_{2j-1}\psi_{2j}\bar{\psi}_{2j+1}, \bar{\psi}_{2k-1}\psi_{2k}\bar{\psi}_{2k+1}\} = 0 \\
\{Q, Q^{\dagger}\} &= H = \sum_{j,k=1}^{N} \{\psi_{2j+1}\bar{\psi}_{2j}\psi_{2j-1}, \bar{\psi}_{2k-1}\psi_{2k}\bar{\psi}_{2k+1}\} \\
&= \sum_{j=1}^{N} \{\bar{\psi}_{2j}\psi_{2j-1}\psi_{2j+2}\bar{\psi}_{2j+3} + \bar{\psi}_{2j-1}\psi_{2j}\psi_{2j+3}\bar{\psi}_{2j+2} \\
&\quad + \bar{\psi}_{2j}\psi_{2j}\psi_{2j+1}\bar{\psi}_{2j+1} + \bar{\psi}_{2j-1}\psi_{2j-1}\psi_{2j}\bar{\psi}_{2j} - \bar{\psi}_{2j-1}\psi_{2j-1}\psi_{2j+1}\bar{\psi}_{2j+1}\}
\end{aligned} \tag{3.2}$$

and thus we have again a realization of the algebra (2.1) and (2.2). The variations are given by

$$\begin{aligned}
\delta\psi_{2k-1} &= -\mathrm{i}^{-1} \sum_{j=1}^{N} \{\bar{\psi}_{2j-1}\psi_{2j}\bar{\psi}_{2j+1}, \psi_{2k-1}\}\bar{\zeta} = \mathrm{i}\bar{\psi}_{2j-3}\psi_{2j-2}\bar{\zeta} + \mathrm{i}\psi_{2j}\bar{\psi}_{2j+1}\bar{\zeta} \\
\delta\bar{\psi}_{2k-1} &= \mathrm{i}^{-1}\zeta \sum_{j=1}^{N} \{\psi_{2j+1}\bar{\psi}_{2j}\psi_{2j-1}, \bar{\psi}_{2k-1}\} = -\mathrm{i}\zeta\psi_{2j+1}\bar{\psi}_{2j} - \mathrm{i}\zeta\bar{\psi}_{2j-2}\psi_{2j-3} \\
\delta\psi_{2k} &= \mathrm{i}^{-1}\zeta \sum_{j=1}^{N} \{\psi_{2j+1}\bar{\psi}_{2j}\psi_{2j-1}, \psi_{2k}\} = \mathrm{i}\zeta\psi_{2k+1}\psi_{2k-1} \\
\delta\bar{\psi}_{2k} &= -\mathrm{i}^{-1} \sum_{j=1}^{N} \{\bar{\psi}_{2j-1}\psi_{2j}\bar{\psi}_{2j+1}, \bar{\psi}_{2k}\}\bar{\zeta} = -\mathrm{i}\bar{\psi}_{2k-1}\bar{\psi}_{2k+1}\bar{\zeta}.
\end{aligned} \tag{3.3}$$

To obtain H in conventional notation (see appendix), we make the following substitutions:

$$\psi_j \to \sigma_j^-, \qquad \psi_j \to \sigma_j^+, \qquad j = 1, \ldots, 2N. \tag{3.4}$$

After subtraction of one constant and multiplication by a coupling constant 4g, we get from (3.3):

$$\begin{aligned}
H = g \sum_{j=1}^{N} \{\sigma_{2j-1}^3\sigma_{2j+1}^3 - \sigma_{2j-1}^3\sigma_{2j}^3 - \sigma_{2j}^3\sigma_{2j+1}^3\} \\
+ 4g \sum_{j=1}^{N} \{\sigma_{2j}^+\sigma_{2j-1}^-\sigma_{2j+2}^-\sigma_{2j+3}^+ + \sigma_{2j-1}^+\sigma_{2j}^-\sigma_{2j+3}^-\sigma_{2j+2}^+\}
\end{aligned} \tag{3.5}$$

† I.e.: $\psi_{j+2N} \equiv \psi_j$, $\bar{\psi}_{j+2N} \equiv \bar{\psi}_j$.

‡ They are most conveniently evaluated using

$$\begin{aligned}
\{ABC, DEF\} = -\{A, D\}BECF + A\{B, D\}ECF - D\{E, A\}BCF + DA\{B, E\}CF + AB\{C, D\}EF \\
- ABD\{C, E\}F + DE\{A, F\}BC - DEA\{B, F\}C + DEAB\{C, F\}.
\end{aligned}$$

where boundary terms were neglected as they do not contribute when we let $N \to \infty$. This Hamiltonian is non-diagonal and contains Ising-like interactions (figure 1).

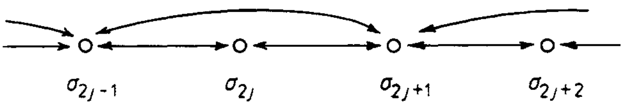

Figure 1. Spin-chain and some of the couplings contained in H.

Remark. Once more the generalization to arbitrary dimension is obvious. One has to put, for example,

$$Q := \sum_{j \in G} \sum_{\hat{n}} \psi_{2j+\hat{n}} \bar{\psi}_{2j} \psi_{2j-\hat{n}}, \qquad Q^{\dagger} = \sum_{j \in G} \sum_{\hat{n}} \bar{\psi}_{2j-\hat{n}} \psi_{2j} \bar{\psi}_{2j+\hat{n}}. \tag{3.1'}$$

We now illustrate how supersymmetry of the Hamiltonian (3.3) forces certain correlation functions to vanish. If A is any operator we write (Feynman 1972):

$$\langle A \rangle := \frac{\operatorname{Tr} A \, \mathrm{e}^{-\beta H}}{\operatorname{Tr} \mathrm{e}^{-\beta H}} = \frac{\operatorname{Tr} \rho A}{\operatorname{Tr} \rho} \qquad \text{where } \rho = \mathrm{e}^{-\beta H} \tag{3.6}$$

The following lemma is easily proved using cyclicity of Tr.

Lemma. If for a symmetry transformation $U[H, U] = 0$, then

$$\langle U^{-1} A U \rangle = \langle A \rangle. \tag{3.7}$$

Taking $U = \mathrm{e}^{\mathrm{i}\zeta Q}$ we get from the lemma

$$\langle \delta \bar{\psi}_{2j-1} \rangle = \langle -\mathrm{i}\zeta \psi_{2j+1} \bar{\psi}_{2j} - \mathrm{i}\zeta \bar{\psi}_{2j-2} \psi_{2j-3} \rangle = 0 \tag{3.8}$$

$$\langle \delta \psi_{2j} \rangle = \langle \mathrm{i}\zeta \psi_{2j+1} \psi_{2j-1} \rangle = 0 \tag{3.9}$$

and therefore

$$\langle \psi_{2j+1} \bar{\psi}_{2j} \rangle = + \langle \psi_{2j-3} \bar{\psi}_{2j-2} \rangle \dagger \tag{3.10}$$

$$\langle \psi_{2j+1} \psi_{2j-1} \rangle = \langle \bar{\psi}_{2j+1} \bar{\psi}_{2j-1} \rangle = 0. \tag{3.11}$$

By choosing $A = \psi_j \psi_k, \psi_j \psi_k \psi_l$, etc, one constructs further identities connecting various correlation functions of higher order; to obtain a restriction on $\langle A_1 \ldots A_n \rangle$, $A_i \in \{\psi_j, \bar{\psi}_j, j = 1, \ldots, 2N\}$ one has to consider the variation of the $(n-1)$-fold product $A'_1 \ldots A'_{n-1}$. Using the substitution (for $1 < j, j+1 < N$ (Lieb *et al* 1961)):

$$\begin{aligned} \psi_{2j+1} \psi_{2j-1} &= \exp[\tfrac{1}{2}\mathrm{i}\pi(\sigma^3_{2j} + 1)] \sigma^-_{2j+1} \sigma^-_{2j-1} \\ \bar{\psi}_{2j-1} \bar{\psi}_{2j+1} &= \exp[-\tfrac{1}{2}\mathrm{i}\pi(\sigma^3_{2j} + 1)] \sigma^+_{2j-1} \sigma^+_{2j+1} \end{aligned} \tag{3.12}$$

one derives from (3.11) the following identities:

$$\langle \exp[\tfrac{1}{2}\mathrm{i}\pi(\sigma^3_{2j} + 1)] \sigma^-_{2j+1} \sigma^-_{2j-1} \rangle = \langle \exp[-\tfrac{1}{2}\mathrm{i}\pi(\sigma^3_{2j} + 1)] \sigma^+_{2j+1} \sigma^+_{2j-1} \rangle = 0. \tag{3.13}$$

As all identities hold independently of N they remain unaffected when one takes $N \to \infty$, i.e. the thermodynamic limit.

† Taking complex conjugates, one gets a similar identity.

4. The Lagrangian approach

In this section we shall utilize the Lagrangian approach to realize the algebra (2.1), (2.2) and thereby gain more insight into its meaning; this approach has the advantage that it yields all possible representations of the basic algebra. We introduce the concept of a superfield $\phi(t, \theta, \bar{\theta})$, where $\theta, \bar{\theta}$ are totally anticommuting quantities ($t \in \mathbb{R}$):

$$\phi(t, \theta, \bar{\theta}) = A(t) + \theta\psi(t) + \bar{\psi}(t)\bar{\theta} + \theta\bar{\theta}F(t); \qquad A = A^*, F = F^*. \tag{4.1}$$

Representing H through the substitution

$$[H, \phi] = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\phi \tag{4.2}$$

we extract from (2.6) a representation of $Q, Q^\dagger$ in terms of differential operators†:

$$Q = -\mathrm{i}\frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\frac{\mathrm{d}}{\mathrm{d}t} \qquad Q^\dagger = \mathrm{i}\frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}\theta\frac{\mathrm{d}}{\mathrm{d}t}. \tag{4.3}$$

Under an infinitesimal transformation

$$\begin{aligned}\delta\phi = \mathrm{i}[\zeta Q + Q^\dagger\bar{\zeta}, \phi] = \zeta\psi + \zeta\bar{\theta}F + \bar{\psi}\bar{\zeta} + \theta\bar{\zeta}F \\ + \tfrac{1}{2}\mathrm{i}\theta\bar{\zeta}\dot{A} - \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\bar{\psi}}\bar{\zeta} - \tfrac{1}{2}\mathrm{i}\zeta\bar{\theta}\dot{A} + \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\zeta\dot{\psi}\end{aligned} \tag{4.4}$$

from which we read off the transformation rules of the components

$$\begin{aligned}&\delta A = \zeta\psi + \bar{\psi}\bar{\zeta} \qquad && \delta\psi = \bar{\zeta}(F + \tfrac{1}{2}\mathrm{i}\dot{A}) \\ &\delta\bar{\psi} = (F - \tfrac{1}{2}\mathrm{i}\dot{A})\zeta \qquad && \delta F = \tfrac{1}{2}\mathrm{i}(\zeta\dot{\psi} - \dot{\bar{\psi}}\bar{\zeta}).\end{aligned} \tag{4.5}$$

The F component transforms like a total derivative and may therefore be used to construct an invariant action integral $\int_{-\infty}^{+\infty} F(t)\,\mathrm{d}t$ as is usually done in supersymmetric theories‡. Other invariants may be constructed if one uses covariant derivatives. These are‡

$$D = \frac{\partial}{\partial\theta} + \tfrac{1}{2}\mathrm{i}\bar{\theta}\frac{\mathrm{d}}{\mathrm{d}t} \qquad \text{and} \qquad \bar{D} = -\frac{\partial}{\partial\bar{\theta}} - \tfrac{1}{2}\mathrm{i}\theta\frac{\mathrm{d}}{\mathrm{d}t}. \tag{4.6}$$

We are now able to reconstruct the simple model of § 2 for $l = 0$ ((2.9), (2.10)). From

$$\begin{aligned}D\phi &= \psi + \bar{\theta}(F + \tfrac{1}{2}\mathrm{i}\dot{A}) - \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\psi} \\ \bar{D}\phi &= \bar{\psi} + \theta(F - \tfrac{1}{2}\mathrm{i}\dot{A}) + \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\bar{\psi}}\end{aligned} \tag{4.7}$$

one obtains

$$\bar{D}\phi D\phi|_F = F^2 + \tfrac{1}{4}\dot{A}^2 + \tfrac{1}{2}\mathrm{i}\dot{\bar{\psi}}\psi - \tfrac{1}{2}\mathrm{i}\bar{\psi}\dot{\psi} \tag{4.8}$$

which, by partial integration, is equivalent to

$$\bar{D}\phi D\phi|_F = F^2 + \tfrac{1}{4}\dot{A}^2 + \mathrm{i}\dot{\bar{\psi}}\psi. \tag{4.9}$$

Another invariant is obtained from

$$\phi^2|_F = 2AF - 2\bar{\psi}\psi. \tag{4.10}$$

† For properties of operators like $\partial/\partial\theta$ etc, see Berezin (1966).

‡ For example, Wess and Zumino (1974), Ferrara *et al* (1974), Wess (1974) and further references quoted therein.

Adding together (4.10) and (4.9) we get a Lagrangian

$$\mathscr{L} = 2F^2 + \tfrac{1}{2}\dot{A}^2 + 2\mathrm{i}\dot{\bar{\psi}}\psi + 2(AF - \bar{\psi}\psi). \tag{4.11}$$

The equations of motion lead to the elimination of F†:

$$\mathscr{L} = \tfrac{1}{2}\dot{A}^2 + 2\mathrm{i}\dot{\bar{\psi}}\psi - \tfrac{1}{2}A^2 - 2\bar{\psi}\psi. \tag{4.12}$$

The corresponding Hamiltonian is

$$H = \tfrac{1}{2}\dot{A}^2 + \tfrac{1}{2}A^2 + 2\bar{\psi}\psi \tag{4.13}$$

and the equivalence with (2.10) follows immediately upon replacing $\psi \to (1/\sqrt{2})\psi$ and A, $\dot{A}$ by their corresponding creation and annihilation operators. The equivalence is established completely if one introduces a lattice index j and a superfield $\phi_j(t, \theta, \bar{\theta})$ at each lattice point. Then (4.13) reads

$$H = \sum_{j=1}^{N} (\tfrac{1}{2}\dot{A}_j^2 + \tfrac{1}{2}A_j^2 + 2\bar{\psi}_j\psi_j). \tag{4.13$'$}$$

For arbitrary l the correct transformation rules are obtained with $A_j \to A_{j+l}$. Interactions not discussed in § 2 may be obtained from $\phi_j\phi_k\phi_l|_F$. They contain anharmonic terms and interactions between spin and harmonic oscillator. It is, however, impossible to construct a pure spin–spin interaction from (4.1).

5. Supersymmetric spin system in the Lagrangian approach

We introduce an *anti*commuting superfield $X(t, \theta, \bar{\theta})$:

$$X(t, \theta, \bar{\theta}) = \chi(t) + \theta a(t) + \bar{\theta} b(t) + \theta\bar{\theta}\phi(t)‡. \tag{5.1}$$

The constrained superfields are

$$\begin{aligned} \bar{D}X = 0: &\qquad X(t, \theta, \bar{\theta}) = \chi + \theta a + \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\chi} \\ DX = 0: &\qquad X'(t, \theta, \bar{\theta}) = \bar{\chi} + \bar{\theta} b - \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\bar{\chi}} \end{aligned} \tag{5.2}$$

and they transform as

$$\delta\chi = \zeta a, \qquad \delta a = -\mathrm{i}\dot{\chi}\bar{\zeta}; \qquad \delta\bar{\chi} = \bar{\zeta} b, \qquad \delta b = \mathrm{i}\zeta\dot{\bar{\chi}}. \tag{5.3}$$

We notice that a and b terms also yield invariant action integrals§. To reconstruct the model of § 3 we give each superfield X a lattice index $j = 1, \ldots, 2N$ and choose $X'_j = X_j^*$; $\bar{\chi}_j = \chi_j^*$, $b_j = a_j^*$. The kinetic invariant is (in almost complete analogy with (4.9)):

$$X_j X_j^*|_\phi = a_j^* a_j + \tfrac{1}{2}\mathrm{i}\dot{\chi}_j\bar{\chi}_j - \tfrac{1}{2}\mathrm{i}\chi_j\dot{\bar{\chi}}_j \cong a_j^* a_j + \mathrm{i}\dot{\chi}_j\bar{\chi}_j. \tag{5.4}$$

The other invariants are constructed from $X_j X_k X_l|_a$ etc; note that $X_j X_k X_l = 0$ if any two of the indices occurring in a product are the same because the superfields anticommute.

† F is therefore called an auxiliary variable.

‡ Here a, b are *not* annihilation and creation operators.

§ The other invariant is of course the ϕ term of (5.1).

To obtain the model of § 3 we have to slightly readjust our notation: we write

$$\begin{aligned} X_{2j} &= \psi_{2j} + \theta a_{2j} + \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\psi}_{2j} \\ X_{2j+1} &= \bar{\psi}_{2j+1} + \theta a^*_{2j+1} + \tfrac{1}{2}\mathrm{i}\theta\bar{\theta}\dot{\bar{\psi}}_{2j+1} \qquad j = 1, \ldots, N\dagger. \end{aligned} \tag{5.5}$$

As the interaction we take

$$\mathscr{L}_{\mathrm{int}} = \sum_{j=1}^{N} \{X_{2j-1}X_{2j}X_{2j+1}|_a + X^*_{2j+1}X^*_{2j}X^*_{2j-1}|_a\}. \tag{5.6}$$

Written out this is

$$\begin{aligned} \mathscr{L}_{\mathrm{int}} = \sum_{j=1}^{N} (&a^*_{2j-1}\psi_{2j}\bar{\psi}_{2j+1} + a_{2j}\bar{\psi}_{2j-1}\bar{\psi}_{2j+1} + \bar{\psi}_{2j-1}\psi_{2j}a^*_{2j+1} + a_{2j-1}\psi_{2j+1}\bar{\psi}_{2j} \\ &+ a^*_{2j}\psi_{2j+1}\psi_{2j-1} + \bar{\psi}_{2j}\psi_{2j-1}a_{2j+1}). \end{aligned} \tag{5.7}$$

Adding this to (5.4) and eliminating the auxiliary variables a_j, a_j^* leads to

$$\begin{aligned} \mathscr{L} = \sum_{j=1}^{N} &(\mathrm{i}\dot{\psi}_{2j}\bar{\psi}_{2j} + \mathrm{i}\dot{\bar{\psi}}_{2j+1}\psi_{2j+1} - \bar{\psi}_{2j-1}\psi_{2j}\psi_{2j+3}\bar{\psi}_{2j+2} - \bar{\psi}_{2j}\psi_{2j-1}\psi_{2j+2}\bar{\psi}_{2j+3} \\ &- \bar{\psi}_{2j-1}\psi_{2j-1}\bar{\psi}_{2j+1}\psi_{2j+1} - \bar{\psi}_{2j-1}\psi_{2j-1}\psi_{2j}\bar{\psi}_{2j} - \bar{\psi}_{2j+1}\psi_{2j+1}\psi_{2j}\bar{\psi}_{2j}) \\ = \sum_{j=1}^{N} &(\mathrm{i}\dot{\psi}_{2j}\bar{\psi}_{2j} + \mathrm{i}\dot{\bar{\psi}}_{2j+1}\psi_{2j+1} - \bar{\psi}_{2j-1}\psi_{2j}\psi_{2j+3}\bar{\psi}_{2j+2} - \bar{\psi}_{2j}\psi_{2j-1}\psi_{2j+2}\bar{\psi}_{2j+3} \\ &+ \bar{\psi}_{2j-1}\psi_{2j-1}\psi_{2j+1}\bar{\psi}_{2j+1} - \bar{\psi}_{2j-1}\psi_{2j-1}\psi_{2j}\bar{\psi}_{2j} - \bar{\psi}_{2j}\psi_{2j}\psi_{2j+1}\bar{\psi}_{2j+1}) \end{aligned} \tag{5.8}$$

which is exactly the Lagrangian analogue of (3.2)‡.

6. Conclusion

Having demonstrated that supersymmetry can be defined for entirely non-relativistic systems and having illustrated its utility in this context we should like to point out that although it will hardly be possible to improve on the simplicity of equations (2.1) and (2.2), it may well be possible to find more refined realizations of it, perhaps yielding—amongst others—Heisenberg-type (anisotropic) interactions. Work along these lines is in progress.

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† I.e. for odd indices we have interchanged the variables with their complex conjugates.

‡ Note that all variables are treated as totally anticommuting.

Appendix

We collect here the formulae connecting Pauli operators and Fermi operators (Lieb *et al* 1961)

$$\psi_j = \exp\left(\tfrac{1}{2}\mathrm{i}\pi \sum_{l=1}^{j-1} (\sigma_l^3 + 1)\right)\sigma_j^-$$
$$\bar{\psi}_j = \sigma_j^+ \exp\left(-\tfrac{1}{2}\mathrm{i}\pi \sum_{l=1}^{j-1} (\sigma_l^2 + 1)\right) \tag{A.1}$$

where $\sigma_j^\pm := \sigma_j^1 \pm \mathrm{i}\sigma_j^2$. Then $\{\psi_j, \psi_k\} = \{\bar{\psi}_j, \bar{\psi}_k\} = 0$; $\{\psi_j, \bar{\psi}_k\} = \delta_{jk}$ implies and is implied by $[\sigma_j^\pm, \sigma_k^\pm] = 0$ for $j \neq k$ and $\{\sigma_j^-, \sigma_j^-\} = \{\sigma_j^+, \sigma_j^+\} = 0$; $\{\sigma_j^-, \sigma_j^+\} = 1$.

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